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LETTER TO THE EDITOR

# Rigorous exponent inequalities for random walks

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**Abstract.** The exponents for the resistance of a random walk path and the 'random walk on a random walk' problem are related to a number of other exponents for random walks. Some rigorous inequalities for these exponents are then established.

Let $S$ be a simple random walk starting at the origin in $\mathbb{Z}^d$. For each $n$, we may consider $S[0, n]$ to be a subgraph of the integer lattice; more precisely, we let $\Gamma_n = (V_n, E_n)$ be the graph with

$$V_n = \{S_t : 0 \leq t \leq n\} \qquad E_n = \{\{S_t, S_{t+1}\} : 0 \leq t < n\}.$$

A number of papers (see Dekeyser *et al* 1987, Manna *et al* 1989 and references therein) have discussed the 'fractal nature' of the graph $\Gamma_n$. Two particular quantities of interest have been the effective resistance between 0 and $S_n$ (assuming a unit resistance on each edge of $\Gamma_n$) and the mean-square displacement of another random walk restricted to lie on $\Gamma_n$. While numerical studies and conjectures have been made about these quantities, no rigorous mathematical statements have been made. In this letter we would like to define appropriate exponents and state what can be said rigorously about them.

The five exponents we discuss are the intersection exponent, the chemical or percolation exponent, the (chronological) loop-erasing exponent, the resistance exponent, and the exponent for mean-square displacement of a random walk on a random walk. All of these exponents have the following properties: they are dimension dependent; the value of the exponent is the same for all $d > 4$; logarithmic corrections appear in $d = 4$; and non-trivial values are taken on for $d = 2, 3$. As mentioned above, the last two exponents were introduced in the physics literature; the intersection exponent has been studied in a number of papers in both the physics and mathematical literature; the chemical or percolation exponent has been studied by Sahimi *et al* (1984) and Movshovitz and Havlin (1988) and is a random walk analogue of the percolation dimension for Brownian motion (Burdzy 1989); and the loop-erasing exponent has been studied in Lawler (1980, 1986, 1988) where the self-avoiding random walk derived from erasing loops is analysed.

Non-trivial estimates have been given for the intersection exponent and the loop-erasing exponent. The purpose of this letter is to show how these estimates give non-trivial estimates on the other exponents. Throughout this letter, $c$ will denote an arbitrary positive constant which depends only on dimension and which may change from line to line. If $f$ and $g$ are two functions of $Z$ or $R$, we write $f \sim g$ if they are

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asymptotic, i.e. $\lim_{n\to\infty}(f(n)/g(n))=1$; $f\asymp g$ if there exist positive constants $c_1$, $c_2$ with $c_1g(n)\leqslant f(n)\leqslant c_2g(n)$; and $f\approx g$ if $\log f\sim\log g$.

Let

$$f(n)=P\{S[0,n]\cap S[n+1,2n]=\varnothing\}.$$

For $d\geqslant 5, f(n)\geqslant c>0$. We define the intersection exponent $\zeta=\zeta_d$ by

$$f(n)\approx\begin{cases}(\log n)^{-\zeta} & d=4\\ n^{-\zeta} & d=2,3.\end{cases}$$

(We have defined the exponent for $d<4$ to be the power of $n$ in the formula, and for $d=4$ we have defined it to be the power of the logarithm. We have also chosen the signs so that the exponent is positive. We will do similarly for the other exponents.) It is known (Lawler 1985) that

$$\zeta_4=\tfrac{1}{2}. \tag{1}$$

For $d<4$ it has been proved that the exponent is well defined and the best rigorous estimates are (Burdzy and Lawler 1989a, b, Burdzy *et al* 1989)

$$\tfrac{1}{4}\leqslant\zeta_3<\tfrac{1}{2} \tag{2}$$

$$\tfrac{1}{2}+(1/8\pi)\leqslant\zeta_2<\tfrac{3}{4}. \tag{3}$$

Duplantier and Kwon (1988) have conjectured from a conformal invariance argument that $\zeta_2=\frac{5}{8}$; Monte Carlo simulations (Duplantier and Kwon 1988, Burdzy *et al* 1989) indicate that this value is not far from the true value. For $d=3$, Monte Carlo simulations suggest $0.28\leqslant\zeta_3\leqslant 0.29$.

We will call a time $t<n$ a cut point for $\Gamma_n$ if $S[0,t]\cap S[t+1,n]=\varnothing$. Note that if $t$ is a cut point, then removal of the edge $\{S_t, S_{t+1}\}$ disconnects $\Gamma_n$. Let $I_{t,n}$ be the indicator function of the event $\{t$ is a cut point for $\Gamma_n\}$ and

$$L_n=\sum_{t=0}^{n-1}I_{t,n}.$$

Since

$$f(t\vee(n-t))\leqslant\langle I_{t,n}\rangle\leqslant f(t\wedge(n-t))$$

we get

$$\langle L_n\rangle\sim cn \qquad d\geqslant 5$$

$$n^{-1}\langle L_n\rangle\approx(\log n)^{-1/2} \qquad d=4 \tag{4}$$

$$\langle L_n\rangle\approx n^{1-\zeta} \qquad d=2,3. \tag{5}$$

Here we use $\langle\ \rangle$ to denote expectation (we will reserve the $E$ notation for expected values of a random walk on a random walk).

If $G=(V,E)$ is any connected graph, the distance between vertices on the graph, $d(x,y)$, is defined to be the length of the shortest (necessarily self-avoiding) path in $G$ connecting $x$ and $y$. For the graph $\Gamma_n$ we define

$$D_n=d(0,S_n).$$

It is clear that if $t$ is a cut point for $\Gamma_n$ then any path from 0 to $S_n$ must contain the edge $\{S_t, S_{t+1}\}$. Therefore

$$L_n\leqslant D_n\leqslant n. \tag{6}$$

If follows immediately that $\langle D_n\rangle \asymp n$ for $d \geqslant 5$. Any subpath of $S[0, n]$ from 0 to $S_n$ can be combined with any subpath of $S[n, n+m]$ from $S_n$ to $S_{n+m}$ to get a subpath of $S[0, n+m]$ from 0 to $S_{n+m}$. We therefore get the subadditivity relation

$$\langle D_{n+m}\rangle \leqslant \langle D_n\rangle + \langle D_m\rangle.$$

From properties of subadditive functions it follows that $\langle D_n\rangle \sim cn$ for $d \geqslant 5$. We define the chemical or percolation exponent $\delta = \delta_d$ by

$$n^{-1}\langle D_n\rangle \approx (\log n)^{-\delta} \qquad d = 4$$
$$\langle D_n\rangle \approx n^{\delta} \qquad d = 2, 3.$$

We do not actually know that the exponent is well defined. However, we can define $\underline{\delta}$ and $\bar{\delta}$ to be in the lim inf and lim sup of the appropriate quantities, e.g.

$$\underline{\delta}_4 = \liminf_{n\to\infty} \frac{\log\langle D_n\rangle - \log n}{-\log\log n}$$

and similarly for the other quantities. For the remainder of the letter we will define exponents as above and let the reader supply the appropriate lim inf and lim sup definitions. From (4)-(6) we get

$$\bar{\delta}_4 \leqslant \tfrac{1}{2} \qquad \underline{\delta}_d \geqslant 1 - \zeta_d \qquad d = 2, 3.$$

Since the root mean square distance of a random walk is $n^{1/2}$, we get an immediate lower bound of $\underline{\delta} \geqslant \frac{1}{2}$ which is better than the bound above if $d = 2$. Because $\Gamma_\infty$ is the entire lattice for $d = 2$, some have conjectured that $\delta_2 = \frac{1}{2}$; however, $\Gamma_n$ is not the entire lattice so it is not clear whether this conjecture is true.

The procedure of (chronological) loop-erasing produces a self-avoiding path from 0 to $S_n$. The procedure is defined as follows: fix $n$ and let $\rho$ and $\sigma$ be defined by

$$\rho = \inf\{t\colon 1 \leqslant t \leqslant n, \exists s < t \text{ with } S_s = S_t\}.$$

Then let $S^1$ be the path with this loop erased, i.e.

$$S_t^1 = \begin{cases} S_t & 0 \leqslant t \leqslant \sigma \\ S_{t+(\rho-\sigma)} & \sigma \leqslant t \leqslant n - (\rho - \sigma). \end{cases}$$

Clearly $S^1$ is a subpath of $S$ from 0 to $S_n$. If $S^1$ is self-avoiding we stop; otherwise, we perform this procedure on $s^1$. Eventually we will get a self-avoiding subpath of $S$ from 0 to $S_n$ which we denote $\hat{S}$. Let $A_n$ be the number of points in $\hat{S}$. Then it follows immediately that

$$L_n \leqslant D_n \leqslant A_n. \tag{7}$$

It has been proved (Lawler 1980) that almost surely $A_n \sim cn$ for $d \geqslant 5$. We define the loop-erasing exponent $\alpha = \alpha_d$ by

$$n^{-1}\langle A_n\rangle \approx (\log n)^{-\alpha} \qquad d = 4$$
$$\langle A_n\rangle \approx n^{\alpha} \qquad d = 2, 3.$$

Again it is not known whether the exponent exists; however, it has been shown (Lawler 1986, 1988) that

$$\tfrac{1}{3} \leqslant \underline{\alpha}_4 \leqslant \bar{\alpha}_4 \leqslant \tfrac{1}{2}$$
$$\bar{\alpha}_d \leqslant \frac{2+d}{6} \qquad d = 2, 3.$$

It is conjectured that $\alpha_4 = \frac{1}{3}$, but that the inequality for $\alpha_d$ $(d = 2, 3)$ is not sharp. This combines with the lower bounds on the percolation exponent to give

$$\tfrac{1}{3} \leq \underline{\delta}_4 \leq \bar{\delta}_4 \leq \tfrac{1}{2} \qquad \tfrac{1}{2} < \underline{\delta}_3 \leq \bar{\delta}_3 \leq \tfrac{5}{6} \qquad \tfrac{1}{2} \leq \underline{\delta}_2 \leq \bar{\delta}_2 \leq \tfrac{2}{3}.$$

It is easy to give examples of paths with $D_n < A_n$, i.e. such that chronological loop-erasing does not give the shortest path from 0 to $S_n$. We conjecture, but have not proven, that in fact $\delta \neq \alpha$. Guttmann and Buisill 1989 have done Monte Carlo simulations which suggest that $\alpha_2 = \frac{5}{8}$ and $\alpha_3 = 0.81\ldots$.

Assume that a resistance of unit 1 is put on each edge of the graph $\Gamma_n$. Let $R_n$ be the effective resistance from 0 to $S_n$. In general it is hard to compute effective resistance; however, it is easy to see that

$$L_n \leq R_n \leq D_n.$$

Also, resistances satisfy the subadditivity property

$$\langle R_{n+m} \rangle \leq \langle R_n \rangle + \langle R_m \rangle.$$

Therefore, $\langle R_n \rangle \sim cn$ for $d \geq 5$. We define the resistance exponent $\rho = \rho_d$ by

$$n^{-1} \langle R_n \rangle \approx (\log n)^{-\rho} \qquad d = 4$$

$$\langle R_n \rangle \approx n^{\rho} \qquad d = 2, 3.$$

We note that the exponent that we had defined is one half the exponent defined by many authors, e.g. the $\zeta$ in Dekeyser *et al* (1987). It then follows that

$$\tfrac{1}{3} \leq \underline{\rho}_4 \leq \bar{\rho}_4 \leq \tfrac{1}{2} \qquad \tfrac{1}{2} < \underline{\rho}_3 \leq \bar{\rho}_3 \leq \tfrac{5}{6} \qquad \tfrac{1}{4} < \underline{\rho}_2 \leq \bar{\rho}_2 \leq \tfrac{2}{3}.$$

In particular, it is rigorously proved that the exponent is positive in two dimensions. Current conjectures for the actual values of the exponent are $\rho_3 = \frac{2}{3}$ and $\rho_2 = \frac{1}{2}$.

Let $G = (V, E)$ be a finite connected simple graph. A simple random walk on $G$ is the Markov chain $X$ with state space $V$ and transition probability

$$p(x, y) = \begin{cases} 1/v(x) & \text{if } \{x, y\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Here $v(x)$ is the degree of $x$, i.e. the number of vertices adjacent to $x$. It is easy to check that the invariant probability for this Markov chain is

$$\phi(x) = (2|E|)^{-1} v(x). \tag{8}$$

Suppose that a unit resistance is put on each edge in $E$. Let

$$\tau_x = \inf\{t \geq 1 : X_t = x\}.$$

Then the effective resistance between $x$ and $y$, $r(x, y)$, is given by (see, e.g., p 54 of Doyle and Snell 1984)

$$r(x, y) = v(x)[P^x\{\tau_y < \tau_x\}]^{-1}. \tag{9}$$

Let $e(x, y)$ be the expected number of visits to $x$ before hitting $y$ starting at $x$, i.e.

$$e(x, y) = \sum_{j=0}^{\infty} P^x\{X_j = x, \tau_y > j\}.$$

It is standard that

$$e(x, y) = [P^x\{\tau_y < \tau_x\}]^{-1}. \tag{10}$$

Let $\tilde{\tau}_x = \inf\{t \geqslant \tau_y : X_t = x\}$. Since $\tilde{\tau}_x$ is a stopping time, it follows from standard Markov chain results that

$$\phi(x) = \frac{e(x, y)}{E^x(\tilde{\tau}_x)} = \frac{e(x, y)}{E^x(\tau_y) + E^y(\tau_x)}. \tag{11}$$

It follows from (8)-(11) that

$$E^x(\tau_y) + E^y(\tau_x) = 2|E| r(x, y). \tag{12}$$

We now consider the graph $\Gamma_n = (V_n, E_n)$. We are interested in estimating

$$T_n = \langle E^0(\tau_{S_n}) \rangle.$$

By symmetry, $\langle E^0(\tau_{S_n}) \rangle = \langle E^{S_n}(\tau_0) \rangle$; hence by (12)

$$\langle E^0(\tau_{S_n}) \rangle = \langle |E_n| R_n \rangle. \tag{13}$$

It follows immediately that $\langle E^0(\tau_{S_n}) \rangle \asymp n^2$ for $d \geqslant 5$. For $d \leqslant 4$ we define the exponent $\gamma = \gamma_d$ by

$$n^{-2} \langle E^0(\tau_{S_n}) \rangle \approx (\log n)^{-\gamma} \qquad d = 4$$

$$\langle E^0(\tau_{S_n}) \rangle \approx n^\gamma \qquad d = 2, 3.$$

Again it follows almost immediately from (13) and the fact that $|E_n| \approx n$ for $d \geqslant 2$ that

$$\gamma_4 = \rho_4 \qquad \gamma d = 1 + \rho d \qquad d = 2, 3.$$

Of course, we can only rigorously assert the appropriate lim inf and lim sup versions of the above equations. The last equation is sometimes referred to as the Einstein relation.

There is another natural way of defining a random walk on a random walk exponent (see, e.g., Dekeyser *et al* 1987) by considering the behaviour of the mean-squared distance $\langle |X_t|^2 \rangle$. This produces two timescales, $t$ and $n$. For $d \geqslant 3$ and $n = \infty$ we can adapt the above argument to show that the expected time until $|X_t| \geqslant n^{1/2}$ grows like $n^\gamma$, if $d = 3$; $n^2 (\log n)^{-\gamma}$ if $d = 4$; and $n^2$ if $d \geqslant 5$. We certainly expect, although have not rigorously proved, that one can invert these relations to get

$$\langle |X_t|^2 \rangle \approx \begin{cases} t^{1/\gamma} & d = 3 \\ t^{1/2} (\log t)^\gamma & d = 4 \\ t^{1/2} & d \geqslant 5. \end{cases}$$

For $d = 2$, the situation is more delicate since the random walk eventually fills the entire lattice. In this case, for fixed $t$, $\langle |X_t|^2 \rangle$ increases with $n$ to the value $t$ for $n = \infty$. It is not so clear how to relate $\gamma$ to $\langle |X_t|^2 \rangle$ in this case. See Manna *et al* (1989) for a discussion of this phenomenon.

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